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Asymptotic Similarities of Fourier and Riemann Coefficients

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1. INTRODUCTION AND RESULTS

If a function f is Riemann integrable on the unit interval [0, 1], we define the Riemann coefficients of f as the errors

$$r_n(f) = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f.$$

In [2], it was observed that the asymptotic behavior of $r_n(f)$ is quite similar to that of the Fourier coefficient

$$a_n(f) = \int_0^1 f(t)\, e^{-i2\pi nt}\, dt$$

of f. In this note, we give some precise results. In general, it seems that if the periodic extension of f from [0, 1) to the real line $\mathbb{R}$ is "smooth," then the coefficients $r_n(f)$ and $a_n(f)$ have the same rate of convergence to zero; otherwise, it is probably true that $a_n(f)$ converges to zero "faster" than $r_n(f)$ does—in the sense of Theorem 3 below. We first state the following known theorem.

THEOREM A. (1) *If f satisfies a Lipschitz condition of order* α, $0 < \alpha < 1$, *say*

$$|f(x) - f(y)| \leqslant M\,|x - y|^{\alpha}$$

for some $M < \infty$ *and all* $x, y \in [0, 1]$, *then both*

$$r_n(f) = O(1/n^{\alpha}) \quad \text{and} \quad a_n(f) = O(1/n^{\alpha}).$$

(2) *If f is of bounded variation on* [0, 1], *then both*

$$r_n(f) = O(1/n) \quad \text{and} \quad a_n(f) = O(1/n).$$

(3) *If f is absolutely continuous on* [0, 1] *and* $f(0) = f(1)$, *then both*

$$r_n(f) = o(1/n) \quad \text{and} \quad a_n(f) = o(1/n).$$

The proofs of (1) and (2) are similar and easy and can be found in [3, 4]. A proof of (3) was given in [1]. It should be noted that the above rates of convergence are sharp (cf. [2]). Let $C^{\beta}(\mathbb{R})$ be the collection of all β-times continuously differentiable functions on $\mathbb{R}$ with $f(t) = f(1+t)$ for all t. We have the following theorem.

THEOREM 1. *Let $\beta \geqslant 1$ and $f \in C^{\beta-1}(\mathbb{R})$. If $f^{(\beta)}$ is of bounded variation on* [0, 1], *then*

$$r_n(f) = O(1/n^{\beta+1}).$$

If $f^{(\beta)}$ is absolutely continuous on [0, 1], *and $f^{(\beta)}(0) = f^{(\beta)}(1)$, then*

$$r_n(f) = o(1/n^{\beta+1}).$$

It is known that the $a_n(f)$ have the same rates of decay as $r_n(f)$ in the above theorem and these rates cannot be improved (cf. [4]). We will show that the above rates of convergence of $r_n(f)$ to zero are also sharp, as in the following theorem.

THEOREM 2. (1) *There exist an $\epsilon_0 > 0$ and a function $f \in C^{\beta-1}(\mathbb{R})$ such that $f^{(\beta)}$ is of bounded variation on* [0, 1] *and*

$$| n^{\beta+1} r_n(f)| \geqslant \epsilon_0$$

for some sequence $n = n_k \to \infty$.

(2) *Let $\{\epsilon_n\}$ be any sequence of real numbers tending to zero. There exists a function $f \in C^{\beta-1}(\mathbb{R})$ such that $f^{(\beta)}$ is absolutely continuous on* [0, 1], *$f^{(\beta)}(0) = f^{(\beta)}(1)$ and*

$$| n^{\beta+1} r_n(f)| \geqslant \epsilon_n$$

for some sequence $n = n_k \to \infty$.

It is clear that if f is continuous on [0, 1], then

$$| a_n(f)| \leqslant \|f\|_p = \left\{\int_0^1 |f|^p\right\}^{1/p}$$

for all n, where $1 \leqslant p < \infty$. Actually, for $1 \leqslant p \leqslant 2$, we even have the stronger inequality

$$\sum_{n=1}^{\infty} \frac{| a_n(f)|^p}{n^{2-p}} \leqslant C_p \|f\|_p^p$$

for some constant C_p independent of f. However, for Riemann coefficients we have the following theorem.

THEOREM 3. *Let $1 \leqslant p < \infty$. For any positive integer n and any constant $C > 0$, there exists a continuous function f on $[0, 1]$ such that $|r_n(f)| > C\|f\|_p$.*

But for $p = \infty$, it is clear that $|a_n(f)| \leqslant \|f\|_\infty$ and $|r_n(f)| \leqslant 2\|f\|_\infty$.

2. PROOFS OF THE ABOVE RESULTS

Proof of Theorem 1. If $f \in C^{\beta-1}(\mathbb{R})$ and $f^{(\beta)}$ is of bounded variation on $[0, 1]$, then

$$\begin{aligned} a_n(f) &= \int_0^1 f(t)\, e^{-i2\pi nt}\, dt = \frac{1}{(i2\pi n)^\beta} \int_0^1 f^{(\beta)}(t)\, e^{-i2\pi nt}\, dt \\ &= (f^{(\beta)}(0) - f^{(\beta)}(1))/(i2\pi n)^{\beta+1} + \frac{1}{(i2\pi n)^{\beta+1}} \int_0^1 e^{-i2\pi nt}\, df^{(\beta)}(t), \end{aligned} \tag{1}$$

so that $|a_n(f)| \leqslant M/n^{\beta+1}$ for all n, with $M = |f^{(\beta)}(0)| + |f^{(\beta)}(1)| + V(f^{(\beta)})$, where $V(f^{(\beta)})$ denotes the total variation of $f^{(\beta)}$ on $[0, 1]$. Hence,

$$\begin{aligned} r_n(f) &= \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{j}{n}\right) - \int_0^1 f \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{n} \sum_{j=1}^{n} a_k(f)\, e^{i2\pi jk/n} - a_0(f) \\ &= \sum_{k=-\infty}^{\infty} a_{kn}(f) - a_0(f) \\ &= \sum_{k=1}^{\infty} [a_{kn}(f) + a_{-kn}(f)]. \end{aligned} \tag{2}$$

Therefore, by (1)

$$|r_n(f)| \leqslant \frac{2M}{n^{\beta+1}} \sum_{k=1}^{\infty} \frac{1}{k^{\beta+1}}.$$

If, in addition $f^{(\beta)}$ is absolutely continuous on $[0, 1]$ and $f^{(\beta)}(0) = f^{(\beta)}(1)$, then by (2) and (1) we have

$$\begin{aligned} |r_n(f)| &= \left| \sum_{k=1}^{\infty} [a_{kn}(f) + a_{-kn}(f)] \right| \\ &\leqslant \frac{1}{n^{\beta+1}} \sum_{k=1}^{\infty} \frac{2}{(2\pi k)^{\beta+1}} \left| \int_0^1 f^{(\beta)}(t) \cos 2\pi knt\, dt \right|. \end{aligned}$$

Since the integrals inside the absolute value signs tend to zero uniformly on k, we have $n^{\beta+1}r_n(f) \to 0$. This completes the proof of Theorem 1.

Proof of Theorem 2. For $\beta \geqslant 1$, we let

$$f(x) = \sum_{j=0}^{\infty} \frac{(-1)^j \cos[(2j+1)\, 2\pi x]}{(2j+1)^{\beta+1}}.$$

It is easy to show that $f \in C^{\beta-1}(\mathbb{R})$ and that $f^{(\beta)}$ is of bounded variation on $[0, 1]$. Also, it is clear from (2) that $r_{2n}(f) = 0$ and

$$r_{2n+1}(f) = \frac{1}{(2n+1)^{\beta+1}} \sum_{j=0}^{\infty} \frac{(-1)^{j+n}}{(2j+1)^{\beta+1}}.$$

This proves the first part of the theorem.

In the proof of the second part of the theorem, we define as in [2] the "saw-tooth" functions

$$v_n(t) = \sum_{k=1}^{n} \chi_{(k-1/2)/n}(t) - nt,$$

where χ_s is the characteristic function of $[s, 1]$. Let $\{\epsilon_n\}$ be any sequence of real numbers tending to zero. We choose a sequence of positive integers n_k, $n_1 < n_2 < \cdots$, so large that whenever $j < p$, n_p/n_j are odd integers and that for each $k = 1, 2,\ldots$, we have

$$\epsilon_{n_k} \leqslant \frac{1}{2^{k-1}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2\pi j)^{\beta+2}}. \tag{3}$$

As in [2], we define our Lebesgue integrable function g by

$$g(t) = \sum_{j=1}^{\infty} v_{n_j}(t)/2^j.$$

Case (i). We first assume that β is even. Let

$$f(t) = \sum_{m \neq 0} \frac{a_m(g)}{(i2\pi m)^{\beta+1}}\, e^{i2\pi mt},$$

where the constants $a_m(g)$ are the Fourier coefficients of g. It is easy to show that $f \in C^{\beta-1}(\mathbb{R})$ and that $f^{(\beta)}$ is absolutely continuous on $[0, 1]$ and $f^{(\beta)}(0) = f^{(\beta)}(1)$. We now study the behavior of $r_{n_k}(f)$. It is not difficult to show that the Fourier coefficients of v_n are

$$a_j(v_n) = \begin{cases} \dfrac{n}{i2\pi j}\, e^{-ij\pi/n} & \text{if } n \text{ is a factor of } j, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

By (2), we obtain for each k, using the fact that β is even and g is an odd function on $\mathbb{R}$,

$$r_{n_k}(f) = \sum_{j=1}^{\infty} \left\{ \frac{a_{jn_k}(g)}{(jn_k)^{\beta+1}\,(i2\pi)^{\beta+1}} + \frac{a_{-jn_k}(g)}{(-jn_k)^{\beta+1}\,(i2\pi)^{\beta+1}} \right\}$$
$$= \frac{2}{(i2\pi)^{\beta+1}} \frac{1}{n_k^{\beta+1}} \sum_{j=1}^{\infty} a_{jn_k}(g)/j^{\beta+1}.$$

By the definition of g and (4),

$$n_k^{\beta+1} r_{n_k}(f) = \frac{2}{(i2\pi)^{\beta+2}} \sum_{j=1}^{\infty} \frac{1}{j^{\beta+1}} \sum_{s=1}^{\infty} \frac{n_s e^{i\pi j n_k/n_s}}{2^s j n_k}\, \delta\left[\frac{jn_k}{n_s}\right]$$
$$= \frac{2}{(i2\pi)^{\beta+2}} \sum_{s=1}^{\infty} \frac{n_s}{n_k 2^s} \sum_{j=1}^{\infty} \frac{1}{j^{\beta+2}}\, e^{i\pi j n_k/n_s}\, \delta\left[\frac{jn_k}{n_s}\right], \tag{5}$$

where we define $\delta[x]$ to be 1 if x is an integer and 0 otherwise.

Now, if $n_s > n_k$, then $n_s = n_k\mu$. Hence,

$$\sum_{j=1}^{\infty} \frac{1}{j^{\beta+2}}\, e^{i\pi j n_k/n_s}\, \delta\left[\frac{jn_k}{n_s}\right] = \sum_{j=1}^{\infty} \frac{e^{i\pi j/\mu}}{j^{\beta+2}}\, \delta\left[\frac{j}{\mu}\right]$$
$$= \sum_{j=1}^{\infty} (-1)^j/(j\mu)^{\beta+2} < 0.$$

Also, if $n_k \geqslant n_s$, then using the fact that n_k/n_s are odd, we have

$$\sum_{j=1}^{\infty} \frac{1}{j^{\beta+2}}\, e^{i\pi j n_k/n_s}\, \delta\left[\frac{jn_k}{n_s}\right] = \sum_{j=1}^{\infty} \frac{(-1)^j}{j^{\beta+2}} < 0.$$

Therefore, by taking just one term in (5), we have

$$-\frac{(i2\pi)^{\beta+2}}{2}\, n_k^{\beta+1} r_{n_k}(f) > \frac{n_k}{n_k 2^k} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^{\beta+2}},$$

so that by (3),

$$|\, n_k^{\beta+1} r_{n_k}(f)| > \frac{1}{2^{k-1}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2\pi j)^{\beta+2}} \geqslant \epsilon_{n_k}.$$

Case (ii). We now let β be an odd integer. In this case, we use the "conjugate saw-tooth" functions

$$u_n(t) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \cos 2\pi knt}{k},$$

and consider

$$h(t) = \sum_{j=1}^{\infty} u_{n_j}(t)/2^j = \sum_{k=0}^{\infty} a_k(h) \cos 2\pi kt,$$

where the n_j are as chosen above. Since $\| u_n \|_1 \leqslant 1$ for all n, h is a Lebesgue integrable function on [0, 1]. We now define our f by

$$f(t) = \sum_{m=1}^{\infty} \frac{2a_m(h)}{(2\pi m)^{\beta+1}} \cos 2\pi mt.$$

Again, it is easy to see that $f \in C^{\beta-1}(\mathbb{R})$ and that $f^{(\beta)}$ is absolutely continuous on [0, 1] and $f^{(\beta)}(0) = f^{(\beta)}(1)$. By the same proof as in case (i), we also have

$$| n^{\beta+1} r_n(f)| \geqslant \epsilon_n$$

for all $n = n_k$, $k = 1, 2, \ldots$.

Proof of Theorem 3. If the theorem were false, then we could find a constant C_p, $1 \leqslant p < \infty$, such that

$$| r_n(f)| \leqslant C_p \| f \|_p \tag{6}$$

for all functions f continuous on [0, 1]. Consider

$$f(t) = \sum_{k=1}^{m} e^{i2\pi n^k t}.$$

It is clear that $r_n(f) = m$, $\| f \|_2 = m^{1/2}$ and

$$\begin{aligned} \| f \|_p^p &\leqslant \| f \|_\infty^{p-1} \| f \|_1 \\ &\leqslant \| f \|_\infty^{p-1} \| f \|_2 \\ &= m^{p-1} m^{1/2} = m^{p-1/2}. \end{aligned}$$

By (6), we have

$$m \leqslant C_p m^{1-1/2p},$$

which is impossible for large m. This completes the proof of the theorem.

References

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